1. Basics of Information

6.004x Computation Structures
Part 1 – Digital Circuits

What is “Information”?

Information, *n.* Data communicated or received that resolves uncertainty about a particular fact or circumstance.

Example: you receive some data about a card drawn at random from a 52-card deck. Which of the following data conveys the most information? The least?

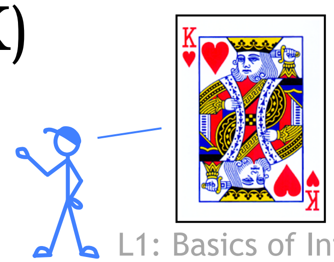
# of possibilities remaining

13 A. The card is a heart

51 B. The card is not the Ace of spades

12 C. The card is a face card (J, Q, K)

1 D. The card is the “suicide king”



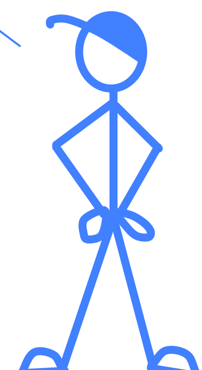
Quantifying Information
(Claude Shannon, 1948)

Given discrete random variable X
- N possible values: $x_1, x_2, \ldots, x_N$
- Associated probabilities: $p_1, p_2, \ldots, p_N$

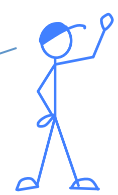
Information received when learning that choice was x_i:

$$I(x_i) = \log_2\left(\frac{1}{p_i}\right)$$

Information is measured in bits (binary digits) = number of 0/1's required to encode choice(s)



$1/p_i$ is proportional to the uncertainty of choice x_i.



Information Conveyed by Data

Even when data doesn't resolve all the uncertainty

$$I(\text{data}) = \log_2\left(\frac{1}{p_{\text{data}}}\right) \qquad \text{e.g., } I(\text{heart}) = \log_2\left(\frac{1}{13/52}\right) = 2 \text{ bits}$$

Common case: Suppose you're faced with N equally probable choices, and you receive data that narrows it down to M choices. The probability that data would be sent is M·(1/N) so the amount of information you have received is

$$I(\text{data}) = \log_2\left(\frac{1}{M \cdot (1/N)}\right) = \log_2\left(\frac{N}{M}\right) \text{ bits}$$

Example: Information Content

Examples:

- information in one coin flip:

  N= **2** M= **1** Info content= $\log_2(2/1) = 1$ **bit**

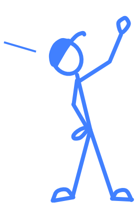
- card drawn from fresh deck is a heart:

  N= **52** M= **13** Info content= $\log_2(52/13) = 2$ **bits**

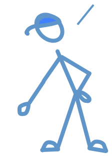
- roll of 2 dice:

  N= **36** M= **1** Info content= $\log_2(36/1) = 5.17$

.17 bits ???

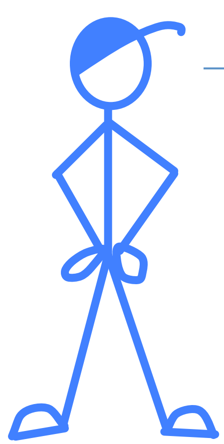


Probability & Information Content



Information content

data	p_{data}	$\log_2(1/p_{data})$
a heart	13/52	2 bits
not the Ace of spades	51/52	0.028 bits
a face card (J, Q, K)	12/52	2.115 bits
the “suicide king”	1/52	5.7 bits



Shannon's definition for information content lines up nicely with my intuition: I get more information when the data resolves more uncertainty about the randomly selected card.

Entropy

In information theory, the entropy H(X) is the average amount of information contained in each piece of data received about the value of X:

$$H(X) = E(I(X)) = \sum_{i=1}^{N} p_i \cdot \log_2\left(\frac{1}{p_i}\right)$$

<u>Example</u>: X={A, B, C, D}

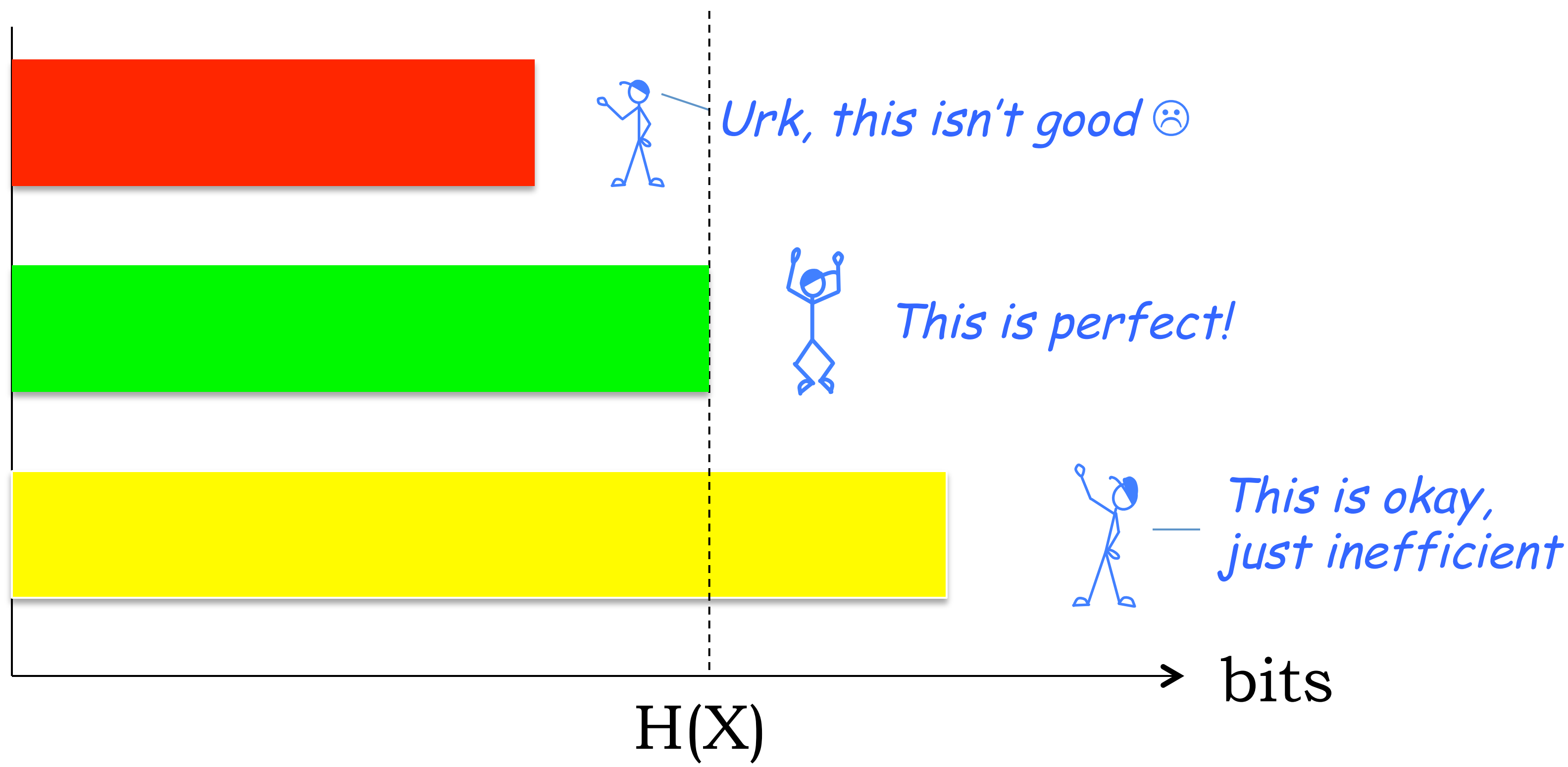
$choice_i$	p_i	$log_2(1/p_i)$
"A"	1/3	1.58 bits
"B"	1/2	1 bit
"C"	1/12	3.58 bits
"D"	1/12	3.58 bits

H(X) = (1/3)(1.58) +
(1/2)(1) +
2(1/12)(3.58)
= 1.626 bits

Meaning of Entropy

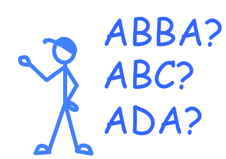
Suppose we have a data sequence describing the values of the random variable X.

Average number of bits used to transmit choice



Encodings

An encoding is an *unambiguous* mapping between bit strings and the set of possible data.

	Encoding for each symbol				Encoding for "ABBA"
	A	B	C	D	
	00	01	10	11	00 01 01 00
	01	1	000	001	01 1 1 01
✗	0	1	10	11	0 1 1 0 

Encodings as Binary Trees

It’s helpful to represent an unambiguos encoding as a binary tree with the symbols to be encoded as the leaves. The labels on the path from the root to the leaf give the encoding for that leaf.

Encoding

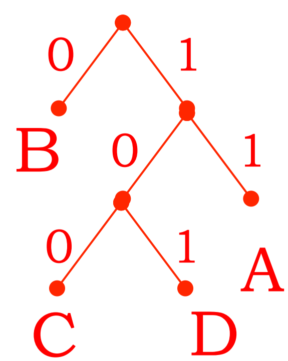
B⟷0

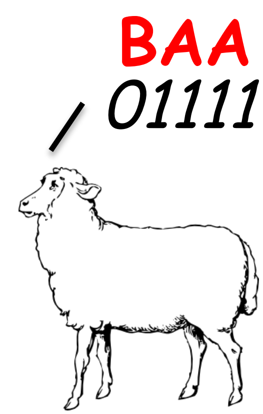
A⟷11

C⟷100

D⟷101

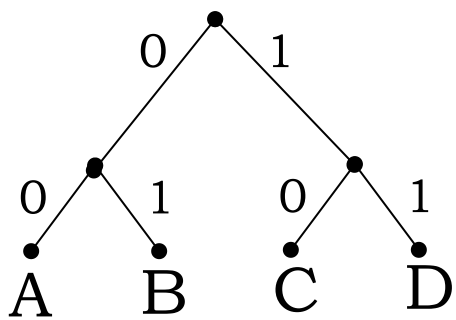
Binary tree

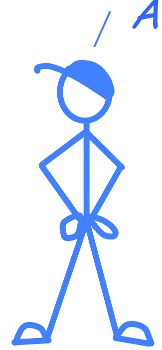




Fixed-length Encodings

If all choices are equally likely (or we have no reason to expect otherwise), then a fixed-length code is often used. Such a code will use at least enough bits to represent the information content.



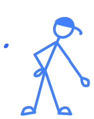


All leaves have the same depth!

Note that the entropy for N equally-probable symbols is

$$\sum_{i=1}^{N}\left(\frac{1}{N}\right)\log_2\left(\frac{1}{\frac{1}{N}}\right) = \log_2(N)$$

Fixed-length are often a little inefficient…



Examples:

- 4-bit binary-coded decimal (BCD) digits **$\log_2(10)=3.322$**
- 7-bit ASCII for printing characters **$\log_2(94)=6.555$**

Encoding Positive Integers

It is straightforward to encode positive integers as a sequence of bits. Each bit is assigned a weight. Ordered from right to left, these weights are increasing powers of 2. The value of an N-bit number encoded in this fashion is given by the following formula:

$$v = \sum_{i=0}^{N-1} 2^i b_i$$

2^{11}	2^{10}	2^9	2^8	2^7	2^6	2^5	2^4	2^3	2^2	2^1	2^0
0	1	1	1	1	1	0	1	0	0	0	0

$$\begin{aligned} V &= 0*2^{11} + 1*2^{10} + 1*2^9 + \ldots \\ &= 1024 + 512 + 256 + 128 + 64 + 16 \\ &= 2000 \end{aligned}$$

Smallest number: **0** Largest number: $2^N - 1$

Hexademical Notation

Long strings of binary digits are tedious and error-prone to transcribe, so we usually use a higher-radix notation, choosing the radix so that it's simple to recover the original bits string.

A popular choice is transcribe numbers in base-16, called hexadecimal, where each group of 4 adjacent bits are represented as a single hexadecimal digit.

Hexadecimal - base 16

0000	0	1000	8
0001	1	1001	9
0010	2	1010	A
0011	3	1011	B
0100	4	1100	C
0101	5	1101	D
0110	6	1110	E
0111	7	1111	F

2^{11}	2^{10}	2^9	2^8	2^7	2^6	2^5	2^4	2^3	2^2	2^1	2^0
0	1	1	1	1	1	0	1	0	0	0	0

$$\underbrace{0111}_{7}\;\underbrace{1101}_{D}\;\underbrace{0000}_{0}$$

0b011111010000 = 0x7D0

Encoding Signed Integers

We use a signed magnitude representation for decimal numbers, encoding the sign of the number (using “+” and “-”) separately from its magnitude (using decimal digits).

We could adopt that approach for binary representations:

N bits

1	1	1	1	1	1	0	1	0	0	0	0

“0” for “+”
“1” for “-”

Range: $-(2^{N-1} - 1)$ to $2^{N-1} - 1$

-2000

But: two representations for 0 (+0, -0) and we’d need different circuitry for addition and subtraction

Two's Complement Encoding

In a two's complement encoding, the high-order bit of the N-bit representation has negative weight:

N bits

-2^{N-1}	2^{N-2}	• • •	• • •	• • •	2^3	2^2	2^1	2^0

"sign bit"

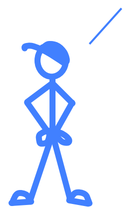
Range: -2^{N-1} to $2^{N-1}-1$

- Negative numbers have "1" in the high-order bit
- Most negative number: 10…0000 $\quad -2^{N-1}$
- Most positive number: 01…1111 $\quad +2^{N-1} - 1$
- If all bits are 1: 11…1111 $\quad -1$
- If all bits are 0: 00…0000 $\quad 0$

More Two's Complement

- Let's see what happens when we add the N-bit values for -1 and 1, keeping an N-bit answer:

$$\begin{array}{r} 11\ldots1111 \\ +\underline{00\ldots0001} \\ 0000000 \end{array}$$



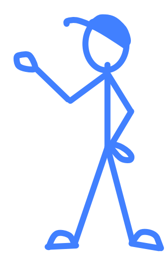
Just use ordinary binary addition, even when one or both of the operands are negative. 2's complement is perfect for N-bit arithmetic!

- To compute B-A, we'll just use addition and compute B+(-A). But how do we figure out the representation for -A?

$$A+(-A) = 0 = 1 + -1$$

$$\begin{aligned} -A &= (-1 - A) + 1 \\ &= \sim A + 1 \end{aligned}$$

$$\begin{array}{r} 1 \\ \underline{-A_i} \\ \sim A_i \end{array}$$



To negate a two's complement value: bitwise complement and add 1.

Variable-length Encodings

We'd like our encodings to use bits efficiently:

> GOAL: When encoding data we'd like to match the length of the encoding to the information content of the data.

On a practical level this means:

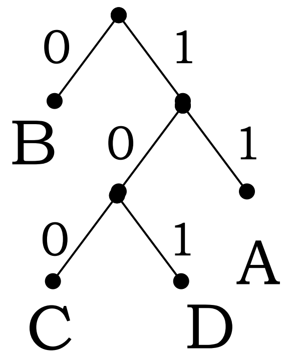
- Higher probability → __shorter__ encodings
- Lower probability → __longer__ encodings

Such encodings are termed variable-length encodings.

Example

$choice_i$	p_i	encoding
"A"	1/3	11
"B"	1/2	0
"C"	1/12	100
"D"	1/12	101

Entropy: H(X) = 1.626 bits



B→0
A→11
C→100
D→101

High probability,
Less information

↕

Low probability,
More information

010011011101

B C A B A D

Expected length of this encoding:

(2)(1/3) + (1)(1/2) + (3)(1/12)(2) = 1.667 bits

Expected length for 1000 symbols:

- With fixed-length, 2 bits/symbol = **2000** bits
- With variable-length code = **1667** bits
- Lower bound (entropy) = **1626** bits

Huffman's Algorithm

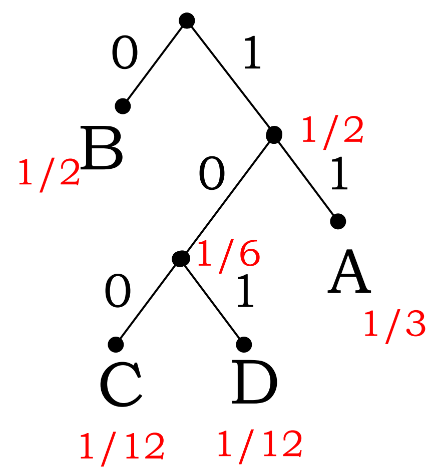
Given a set of symbols and their probabilities, constructs an optimal variable-length encoding.

Huffman's Algorithm:

- Build subtree using 2 symbols with lowest p_i
- At each step choose two symbols/subtrees with lowest p_i, combine to form new subtree
- Result: optimal tree built from the bottom-up

Example:

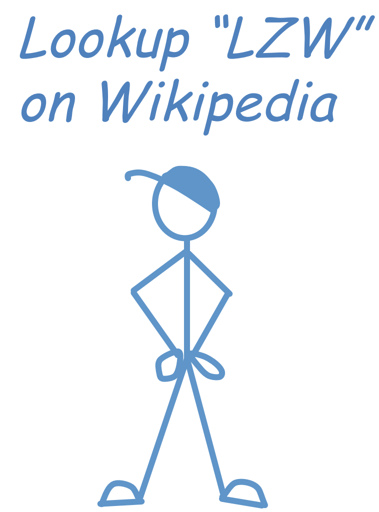
~~A=1/3~~, B=1/2, ~~C=1/12~~, ~~D=1/12~~



Can We Do Better?

Huffman's Algorithm constructed an optimal encoding… does that mean we can't do better?

To get a more efficient encoding (closer to information content) we need to encode sequences of choices, not just each choice individually. This is the approach taken by most file compression algorithms…



AA=1/9,	AB=1/6,	AC=1/36,	AD=1/36
BA=1/6,	BB=1/4,	BC=1/24,	BD=1/24
CA=1/36,	CB=1/24,	CC=1/144,	CD=1/144
DA=1/36,	DB=1/24,	DC=1/144,	DD=1/144

Using Huffman's Algorithm on pairs:

Average bits/symbol = 1.646 bits

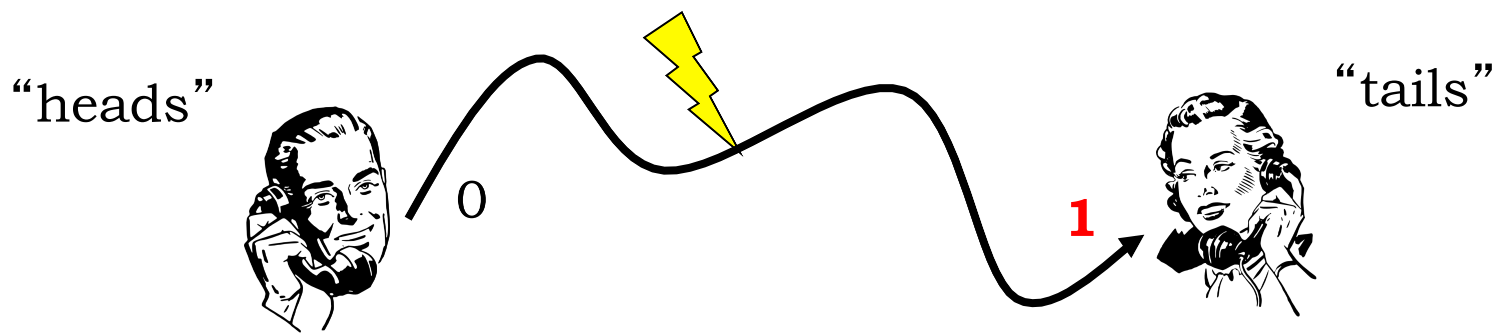
Error Detection and Correction

Suppose we wanted to reliably transmit the result of a single coin flip:

Heads: "0"

Tails: "1"

Further suppose that during processing a single-bit error occurs, i.e., a single "0" is turned into a "1" or a "1" is turned into a "0".

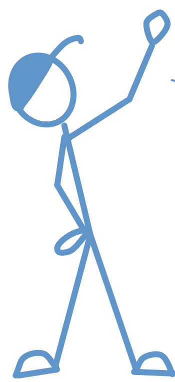


Hamming Distance

HAMMING DISTANCE: The number of positions in which the corresponding digits differ in two encodings of the same length.

0110010

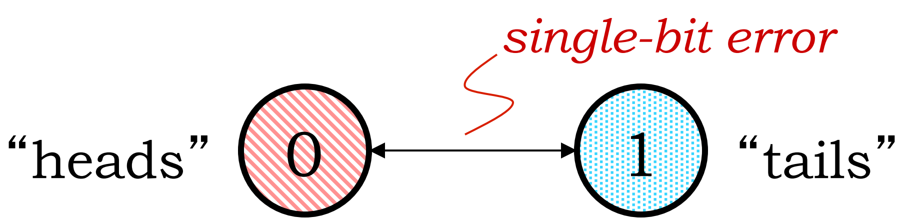
0100110



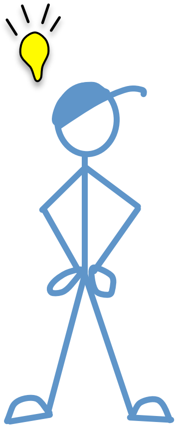
Hamming Distance & Bit Errors

The Hamming distance between a valid binary code word and the same code word with a single-bit error is 1.

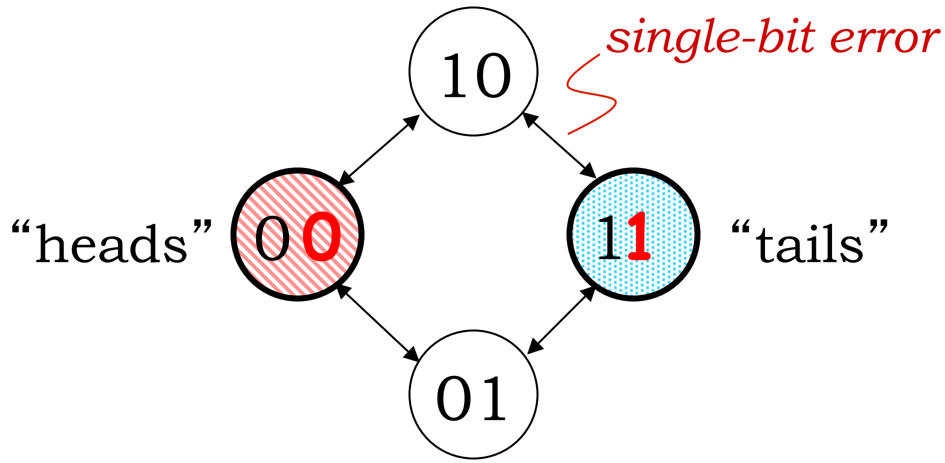
The problem with our simple encoding is that the two valid code words ("0" and "1") also have a Hamming distance of 1. So a single-bit error changes a valid code word into another valid code word…



Single-bit Error Detection



What we need is an encoding where a single-bit error does *not* produce another valid code word.



A parity bit can be added to any length message and is chosen to make the total number of "1" bits even (aka "even parity"). If min HD(code words) = 1, then min HD(code words + parity) = 2.

Parity check = Detect Single-bit errors

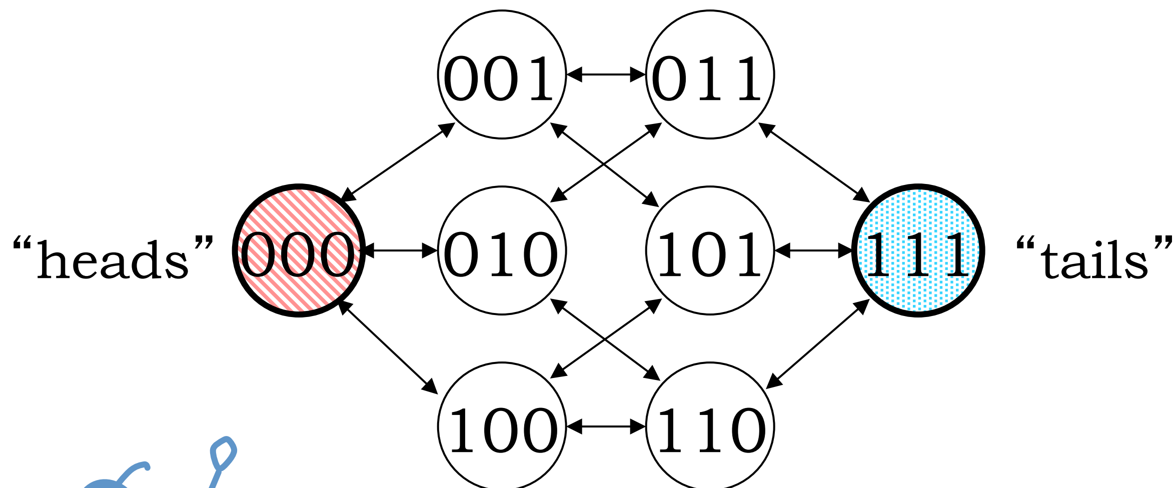
- To check for a single-bit error (actually any odd number of errors), count the number of 1s in the received message and if it's odd, there's been an error.

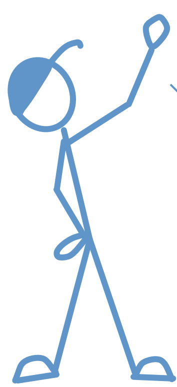
  0 1 1 0 0 1 0 1 0 0 1 1 → original word with parity
  0 1 1 0 0 0 0 1 0 0 1 1 → single-bit error (detected)
  0 1 1 0 0 0 1 1 0 0 1 1 → 2-bit error (not detected)

- One can "count" by summing the bits in the word modulo 2 (which is equivalent to XOR'ing the bits together).

Detecting Multi-bit Errors

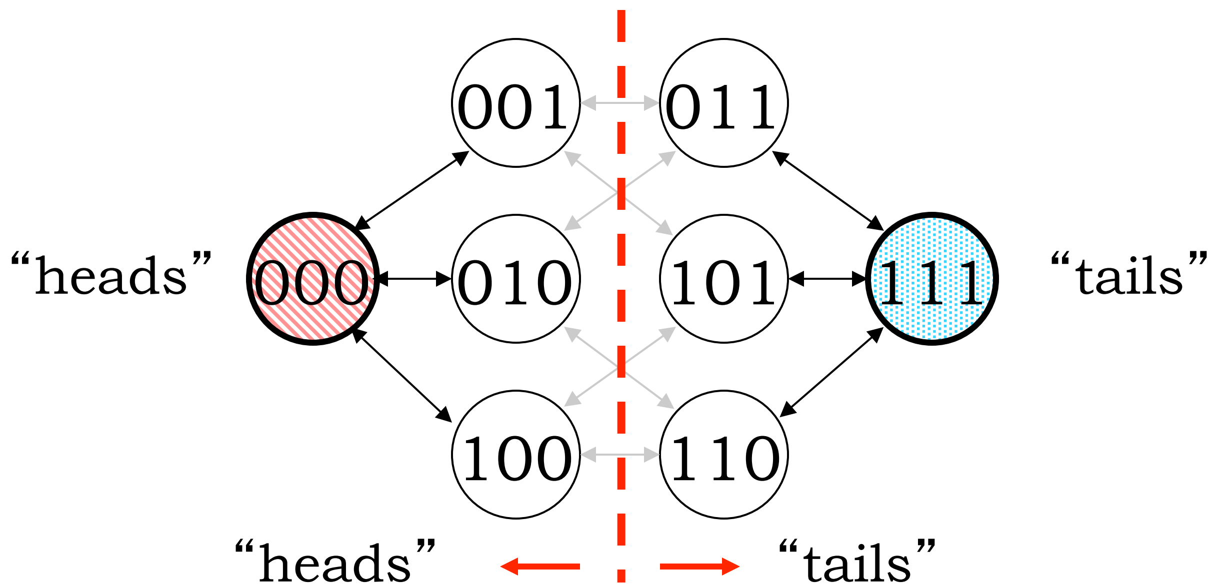
To detect E errors, we need a minimum Hamming distance of E+1 between code words.





With this encoding, we can detect up to two bit errors. Note that HD(000,111) = 3…

Single-bit Error Correction



By increasing the Hamming distance between valid code words to 3, we guarantee that the sets of words produced by single-bit errors don't overlap. So assuming at most one error, we can perform *error correction* since we can tell what the valid code was before the error happened.

To correct E errors, we need a minimum Hamming distance of 2E+1 between code words.

Summary

- Information resolves uncertainty
- Choices equally probable:
  - N choices down to M$\Rightarrow$ $\log_2(N/M)$ bits of information
  - use fixed-length encodings
  - encoding numbers: 2's complement signed integers
- Choices not equally probable:
  - choice_i with probability $p_i \Rightarrow \log_2(1/p_i)$ bits of information
  - average amount of information = $H(X) = \sum p_i \log_2(1/p_i)$
  - use variable-length encodings, Huffman's algorithm
- To detect E-bit errors: Hamming distance > E
- To correct E-bit errors: Hamming distance > 2E

Next time:

- encoding information electrically
- the digital abstraction
- combinational devices